Functional renormalisation group for few-nucleon systems
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Few body systems

- The FRG method has already been applied to few-body problems in a number of papers.
- The Skornyakov–Ter-Martirosyan equation for three-body systems rederived Diehl et al.
- The Efimov effect was addressed by Moroz et al. [Infinite tower of resonances in 3bd, if zero energy bound state in 2bd.]
- Schmidt and Moroz treat the four-body problem in the presence of the three-body Efimov effect (Jaramillo’s talk)
- The process of dimer-dimer scattering was studied in detail in our work (and convergence)
- Low energy physics
Nuclear Physics and EFT

- Effective field theories have been applied in two guises (assumes $p/\Lambda \ll 1$)
  - Pionless and with pions
  - Pionless contact forces with counter terms
- Low-energy scales (deuteron binding $\approx 2\text{MeV}$), scattering lengths, etc. very different from naive QCD scales.
- Formal analysis, using RG methods, has found relevant and irrelevant terms, limit cycle behaviour, etc.
- Can’t deal with many-body problem (nuclei!) in same rigorous manner: Fermi momentum is another low-energy scale!
- Still FRG is one way out, if less rigorous.
The running effective action satisfies an exact flow equation,

\[ \partial_k \Gamma = -\frac{i}{2} \text{Str} \left[ (\partial_k R) (\Gamma^{(2)} - R)^{-1} \right]. \]

where \( \Gamma^{(2)} \) denotes the second functional derivative, and \( R \) is a matrix of cut-off functions.

These functions suppress the contributions of fluctuations with momenta below the running scale \( k \), and drive the evolution of the system as \( k \) is lowered.

The main complexity is in the operation is the supertrace \( \text{Str} \), over both energy-momentum variables and internal indices. It is needed since we consider a mixed system of fermions (nucleons) and bosonic dimers in this work.
the REA is usually truncated to a finite number of local terms. Reduces the flow equation to a set of coupled ODE’s.

Use of local interaction links to effective field theories (Beane, van Kolck):
- The action used to study the Efimov effect in three-boson systems is just the leading terms of the effective field theory.
- If we look at dense matter, power counting arguments cannot be used, and we just try to describe important physics, such as superfluidity.

Natural scaling of the evolution equations for large $k$ (goes to unitary limit of infinite scattering length).
- All the evolution equations collapse to these universal ones.
FRG in this work

- Look at up to four nucleons, intention is to move to larger numbers of nucleons and to nuclear matter.
- Introduce boson fields to describe interacting pairs of nucleons. Links scattering of nucleons in vacuum to pairing in systems of many nucleons.
- Make use of Wigner’s SU(4) “supermultiplet" symmetry.
- Look at the evolution of the REA, without assuming SU(4) symmetry. In the case of exact symmetry there are two decoupled sets of equations.
  - One set same as fermions with spin degrees of freedom only. The other set is identical to case of interacting bosons (Moroz et al).
  - One set contains the Efimov effect and limit-cycle behaviour of the 3-body couplings (P. F. Bedaque, H.-W. Hammer, and U. van Kolck). Other set much simpler.
Phaseshifts

\[ 1S_0 \]
\[ 3S_1 \]
\[ 3P_0 \]
\[ 3P_1 \]
\[ 3P_2 \]
\[ 1D_2 \]
\[ 3D_1 \]

Phase Shift [°] vs. \( E_{\text{lab}} \) [MeV]
Running effective action

- The nuclear force is just strong enough to generate a bound state, the deuteron, in one of the $S$-wave channels.
- Its isospin-1 analogue is only just unbound and appears as a virtual state very close to threshold.
- We thus find it useful to introduce boson fields to describe the lowest two-body states in both these channels.
- We use the notation $t$ for the (spin-triplet) vector-isoscalar boson field, corresponding to the deuteron, and $s$ for the (spin-singlet) scalar-isovector one.
- These we refer to collectively as “dimers”, and label by a capital $D$.
- When specifying quantum numbers of particular channels, we use the $SO(4)$ notation $(S, I)$, where $S$ and $I$ denote the resultant spin and isospin quantum numbers, respectively.
The form for the REA we use is (real time; local; derivative expansion)

\[
\Gamma = \int d^4 p \, \psi_{m_i m_s}^\dagger (p_0, p) \left( p_0 - \frac{p^2}{2M} + i\epsilon \right) \psi_{m_i m_s} (p_0, p) \\
+ \int d^4 p \, t_i^\dagger (p_0, p) \left( Z_{\phi, i} p_0 - Z_{m, i} \frac{p^2}{4M} - u_{1, i} + i\epsilon \right) t_i (p_0, p) \\
+ \int d^4 p \, s_a^\dagger (p_0, p) \left( Z_{\phi, s} p_0 - Z_{m, s} \frac{p^2}{4M} - u_{1, s} + i\epsilon \right) s_a (p_0, p) \\
+ \Gamma_2 + \Gamma_3 + \Gamma_4.
\]

Boson fields should be non-propagating auxiliary fields for \( k \to \infty \). Their wave function and kinetic-mass renormalisation factors (\( Z_{\phi, i} \) and \( Z_{m, i} \)) should tend to zero in this limit.
Bosonisation introduces self-energies for the boson fields \((u_{1,i})\) and couplings of the dimers to pairs of nucleons in the \((1,0)\) and \((0,1)\) two-body channels.

\[
\Gamma_2 = \int \delta^{(4)}(p_1 + p_2 - p_3) d^4 p_1 d^4 p_2 d^4 p_3 \mathcal{V}_2
\]

\[
\mathcal{V}_2 = \frac{1}{2 \sqrt{2}} g_t \left[ t^\dagger(p_{30}, p_3) \cdot \left[ \psi(p_{10}, p_1) \psi^C(p_{20}, p_2) \right]^{(1,0)} + \text{h.c.} \right] + \frac{1}{2 \sqrt{2}} g_s \left[ s^\dagger(p_{30}, p_3) \cdot \left[ \psi(p_{10}, p_1) \psi^C(p_{20}, p_2) \right]^{(0,1)} + \text{h.c.} \right],
\]

where

\[
\psi^C_{m_{t2} m_{s2}} = \tau_{m_{s1} m_{s2}}^2 \sigma_{m_{t1} m_{t2}}^2 \psi_{m_{t1} m_{s1}}.
\]

The values of \(g_i\) depend on a choice of scale in the dimer fields; occur in physics in combinations such as \(g_i^2 / u_{1,i}\). \(g_i\) do not run in vacuum. We choose their values to be equal,

\[
g_{t,s} = g.
\]
To describe the three-nucleon channels, we introduce a set of local dimer-nucleon interactions, described by the term

\[ \Gamma_3 = - \int \delta^{(4)}(p_1 + p_3 - p_2 - p_4) d^4p_1 d^4p_2 d^4p_3 d^4p_4 \mathcal{V}_{3,\text{DN}} \]

\[ \mathcal{V}_{3,\text{DN}} = \sum_{ij=\text{t,s}} \lambda^{(1/2,1/2)}_{ij} \left[ i^\dagger(p_{30}, p_3) \psi^\dagger(p_{10}, p_1) \right]^{(1/2,1/2)} \cdot \left[ \psi(p_{20}, p_2) j(p_{40}, p_4) \right]^{(1/2,1/2)} + \lambda^{(3/2,1/2)}_{tt} \left[ t^\dagger(p_{30}, p_3) \psi^\dagger(p_{10}, p_1) \right]^{(3/2,1/2)} \cdot \left[ \psi(p_{20}, p_2) t(p_{40}, p_4) \right]^{(3/2,1/2)} + \lambda^{(1/2,3/2)}_{ss} \left[ s^\dagger(p_{30}, p_3) \psi^\dagger(p_{10}, p_1) \right]^{(1/2,3/2)} \cdot \left[ \psi(p_{20}, p_2) s(p_{40}, p_4) \right]^{(1/2,3/2)}, \]

where \( \lambda^{(1/2,1/2)}_{ts} = \lambda^{(1/2,1/2)}_{st} \). These have been expressed in terms of interactions in the doublet-doublet channel, with spin-isospin quantum numbers \((1/2, 1/2)\), and the quartet-doublet channels, with quantum numbers \((3/2, 1/2)\) or \((1/2, 3/2)\). The \((1/2, 1/2)\) channel has the quantum numbers of the ground states of \(^3\text{H}\) and \(^3\text{He}\).
Finally, we introduce one class of four-nucleon interactions, represented by local two-body dimer-dimer interactions and described by the term

\[
\Gamma_{2,DD} = - \int \delta^{(4)}(p_1 + p_3 - p_2 - p_4)d^4p_1d^4p_2d^4p_3d^4p_4 \mathcal{V}_{2,DD},
\]

\[
\mathcal{V}_{2,DD} = \frac{1}{2} \sum_{i=t,s} u_{2,i} \left( i^\dagger(p_{10}, p_1) \cdot i(p_{20}, p_2) i^\dagger(p_{30}, p_3) \cdot i(p_{40}, p_4) \right. \\
- \frac{1}{3} i^\dagger(p_{10}, p_1) \cdot i^\dagger(p_{30}, p_3) i(p_{20}, p_2) \cdot i(p_{40}, p_4) \bigg) \\
+ u_{2,ts} \left( t^\dagger(p_{10}, p_1) \cdot t(p_{20}, p_2) s^\dagger(p_{30}, p_3) \cdot s(p_{40}, p_4) \right) \\
+ \frac{1}{12} \sum_{ij=t,s} \bar{u}_{2,ij} i^\dagger(p_{10}, p_1) \cdot i^\dagger(p_{30}, p_3) j(p_{20}, p_2) \cdot j(p_{40}, p_4),
\]

where \( \bar{u}_{2,ts} = \bar{u}_{2,sl} \). The \( u_{2,i} \) terms act in the \((2,0)\) and \((0,2)\) two-dimer (four-nucleon) channels, the \( u_{2,ts} \) terms act in the \((1,1)\) channel, and the \( \bar{u}_2 \) terms act in the \((0,0)\) channel. Only the last of these has the quantum numbers of the ground state of the \( \alpha \) particle.
Asymptotic

- For large $k$ the effects of SU(4)-breaking become negligible and so the evolution equations reduce to the simpler, SU(4)-symmetric form. This allows us to determine the initial conditions.
- Most of the parameters are "irrelevant": physical values independent of the precise values of the initial conditions. Three exceptions:
  - Two of them are the self-energies, $u_{1,i}$, which are "relevant" parameters in RG language, as discussed in Ref. [5]. determine the strength of the interaction between the fermions and can be related either to the $NN$ scattering lengths, $a_t$ and $a_s$, or, in the channel with the bound state, to the deuteron binding energy, $E_d = -\frac{1}{Ma_t^2}$.
  - The third parameter is associated with the spatially symmetric channel that displays the Efimov effect [12] and as a result the evolution of its coupling constant for it has a limit-cycle behaviour at large-$k$. Just as in the analogous EFT treatment [?], one piece of three-body data is needed to fix the starting point on this cycle.
Evolution equations

- **Truncation** $\implies$ choice of point around which we expand our energies.
  - For the dimers choose energy of the lowest two-body bound state, $E_d$.
  - For four nucleons this means that we expand around the deuteron-deuteron threshold.
  - For the nucleons we expand around one half of this binding energy.
  - In nuclear-dimer channel we expand about an energy $E_d/2$ below the scattering threshold.
- **cut-off functions for the nucleons and dimers.** use Litim’s suggestion (no energy dependence)
  
  $$R_N(q; k) = \frac{k^2 - q^2}{2M} \theta(k - q), \quad R_D(q; k) = Z_\phi(k) \frac{k^2 - q^2}{4M} \theta(k - q).$$

  - Note that we have assumed that the two $Z_\phi$’s are equal.
  - Inclusion of $Z_\phi$ in $R_D$ allows us to scale out the parameters $g$, $M$ and $a_t$, leaving dimensionless expressions.
  - For example, a generic three-body coupling $\lambda$ has a natural scaling of the form $\Lambda(\kappa) = \lambda(k)/a_t^2 M g^2$.
  - Analogously a generic four-body coupling $u_2$ can be scaled as $U_2(\kappa) = g^{-4} M^{-3} a_t^{-3} u_2(k)$. 
One-boson terms

- Evolution equations for \( u_{1,i}, Z_{\Phi,i} \) and \( Z_{m,i} \) are relatively straightforward.
- For example, both the \( u_{1,i} \) satisfy the differential equation

\[
\partial_k u_{1,i}(k) = \frac{g^2}{2} \frac{1}{(2\pi)^3} \int d^3q \frac{\partial_k R_N(q;k)}{\left(\frac{q^2}{2M} + R_N(q;k) - \mathcal{E}_d/2\right)^2}.
\]

- \( \mathcal{E}_d \): choice of expansion point, contour-integrated over \( q^0 \).
- Exact differential: solution grows linearly with \( k \) as \( k \to \infty \).
- In the deuteron (t) channel, there is a bound state. Propagator in this channel should therefore have a pole at \( p_0 = 0 \):

\( u_{1,t}(k = 0) = 0 \).
- The solution to that satisfies this can be written in the form

\[
\begin{align*}
 u_{1,t}(k) &= -\frac{g^2 M}{4\pi a_t} - \frac{g^2}{2} \frac{1}{(2\pi)^3} \int d^3q \left[ \frac{1}{\left(\frac{q^2}{2M} + R_N(q) - \mathcal{E}_d/2\right) - \frac{1}{\frac{q^2}{2M}}} \right] \\
 &= \frac{g^2 M}{4\pi a_t} \left( \frac{4}{3\pi} \kappa + \frac{2}{3\pi} \frac{\kappa}{\kappa^2 + 1} + \frac{2}{\pi} \cot^{-1}(\kappa) - 1 \right),
\end{align*}
\]
**S channel**

- The $S$ channel has no bound state and hence a negative scattering length, $a_s$.

- If we expand about zero energy $u_{1,s}(k = 0) = -\frac{g^2 M}{4 \pi a_s}$, which is obtained by replacing $a_t$ by $a_s$ and setting $\mathcal{E}_d$.

- We haven taken the deuteron energy as our expansion point.

- The appropriate boundary condition can be obtained by replacing $a_t$ by $a_s$ in the first part of the right-hand side $u_{1,t}$ while keeping $\mathcal{E}_d$ in the second term.

- Introducing the dimensionless parameter $\alpha$ ($\leq 1$) by $1/a_s = \alpha/a_t$, we can write the solution in the form

$$u_{1,s}(k) = u_{1,t}(k) + \frac{g^2 M}{4 \pi a_t} (1 - \alpha).$$

- This indicates that this approach is at least correct to first order in the strength of SU(4)-breaking, $1 - \alpha$. 
The wave function renormalisation factors \( Z_{\phi,i} \) both satisfy the same equation and we can impose the same boundary condition that they vanish as \( k \to \infty \) on them. They then have the same form in both channels,

\[
Z_{\phi,i}(k) = \frac{1}{4} \frac{1}{(2\pi)^3} \int d^3q \frac{1}{\left[ \frac{q^2}{2M} + R_N(q) - \mathcal{E}_d/2 \right]^2}
\]

\[
= a_i g^2 M^2 \left( \frac{2\kappa (5\kappa^2 + 3)}{3\pi (\kappa^2 + 1)^2} + \frac{2}{\pi} \cot^{-1}(\kappa) \right).
\]

- Cut-off respects Galilean invariance (Birse). Mass same as energy renormalisation factors, \( Z_{m,i} = Z_{\phi,i} \).
- Introduce a compact notation for the inverse propagators,

\[
E_{NR}(q) = \frac{q^2}{2M} - \mathcal{E}_d/2 + R_N(q),
\]

\[
E_{DR,i}(q) = Z_\phi \left[ \frac{q^2}{4M} + u_{1,i}(q)/Z_\phi - \mathcal{E}_d + R_D(q)/Z_\phi \right],
\]

where we have suppressed the implicit \( k \) dependence of the running quantities in these expressions.
Three-body couplings

The evolution equations for the three-body (nucleon-dimer) couplings decouple into a set of four for the \( (1/2, 1/2) \) channel and two separate equations for the \( (3/2, 1/2) \) and \( (1/2, 3/2) \) channels. They have the forms

\[
\partial_k \lambda^{(1/2,1/2)}_{tt} = \lambda^{(1/2,1/2)}_{st} \lambda^{(1/2,1/2)}_{ts} I_{1,t} + \left( \lambda^{(1/2,1/2)}_{tt} \right)^2 I_{1,t} + \frac{1}{16} g^4 \left( I_{3,t} + 9 I_{3,s} \right),
\]

\[
\partial_k \lambda^{(1/2,1/2)}_{ts} = \lambda^{(1/2,1/2)}_{tt} \lambda^{(1/2,1/2)}_{ts} I_{1,t} + \lambda^{(1/2,1/2)}_{ts} \lambda^{(1/2,1/2)}_{ss} I_{1,s} + \frac{1}{4} g^2 \left( \left[ 3 \lambda^{(1/2,1/2)}_{tt} - \lambda^{(1/2,1/2)}_{ts} \right] I_{1,t} + \left[ 3 \lambda^{(1/2,1/2)}_{ss} - \lambda^{(1/2,1/2)}_{ts} \right] I_{1,s} \right) + \frac{3}{16} g^4 \left( I_{3,t} + I_{3,s} \right),
\]

\[
\partial_k \lambda^{(3/2,1/2)}_{tt} = \left( \lambda^{(3/2,1/2)}_{tt} \right)^2 I_{1,t} + \frac{1}{4} g^4 I_{3,t},
\]

where we have displayed only half the equations; the others can obtained by appropriate interchanges of spin and isospin labels, in particular \( t \leftrightarrow s \).
Dimer-dimer couplings

The evolution equations for the four-body (dimer-dimer) couplings $u_2$ or $\bar{u}_2$ also decouple, into a set of four for the $(0,0)$ channel and three separate equations for the $(2,0)$, $(0,2)$ and $(1,1)$ channels. They have the forms (where again we have displayed only half the equations),

\[
\begin{align*}
\partial_k u_{2,t} &= \frac{1}{2} u_{2,t} K_{1,t} - 2 g^2 \lambda_{tt}^{(3/2,1/2)} K_2 - \frac{3}{4} g^4 K_3, \\
\partial_k u_{2,ts} &= \frac{1}{2} u_{2,ts} K_{1,ts} - g^2 \frac{1}{3} \left( 2 \sum_{i=t,s} \lambda_{ii}^{(3/2,1/2)} + \sum_{ij=t,s} \lambda_{ij}^{(1/2,1/2)} \right) K_2 - \frac{3}{4} g^4 K_3, \\
\partial_k \bar{u}_{2,t} &= \frac{1}{4} \bar{u}_{2,t} K_{1,t} + \frac{1}{4} \bar{u}_{2,ts} \bar{u}_{2st} K_{1,s} - 12 g^2 \lambda_{tt}^{(1/2,1/2)} K_2 + \frac{3}{4} g^4 K_3, \\
\partial_k \bar{u}_{2,ts} &= \frac{1}{4} \bar{u}_{2,ts} K_{1,t} + \frac{1}{4} \bar{u}_{2ss} \bar{u}_{2ts} K_{1,s} - 12 g^2 \lambda_{st}^{(1/2,1/2)} K_2 - \frac{9}{4} g^4 K_3.
\end{align*}
\]

Only a limited subset of the thee-body couplings enter: constraints on angular momentum and isospin.
SU(4)-symmetric limit

- We first look at the case of exact supermultiplet symmetry, $a_s = a_t$.
- We can reduce the problem to a limited set of parameters.
- These include two dimensionless three-body coupling constants, one of which (the coupling $\lambda'$ in the $(1/2,1/2)$ channel) exhibits the Efimov effect.
- This effect is a remarkable feature of any three-body system with an attractive short-range interaction.
- In the unitary limit ($a \to \infty$) such a system possesses an infinite number of three-body bound states with a geometric spectrum.
- Even away from the unitary limit, these systems display universal features, such as relations between various three- and four-body observables [6, 7].
- In the framework of a renormalisation group, the evolution of the corresponding three-body force shows a limit-cycle behaviour.
- Away from the unitary limit, the finite inverse scattering length provides an infrared cut-off on the Efimov behaviour.
- This leaves a unique shallowest bound state which, in the nuclear context, we interpret as the triton.
- As a result, the periodic behaviour of the three-body coupling stops when the cut-off scale decreases to a value comparable with $1/a$ where it becomes almost independent of the running scale.
Compare the asymptotic (unitary limit) results for $\Lambda'$ to the numerical solution of the full equation. This coupling diverges periodically in $t = \ln(\kappa) = \ln(ka_t)$: the geometrically spaced states of the Efimov effect.

An example of the full evolution of $\Lambda'$ (solid blue line) compared to the asymptotic evolution (red dashed line). The full evolution was started by integrating downwards from $t = 20$, with the same initial condition as chosen for the asymptotic solution.

The linear growth seen for large $t$ is the signal of Efimov behaviour. At $t \simeq 0$, the scale $1/a_t$ becomes important and acts as a low-energy cut-off.
Complex parts

If we add the appropriate $i\epsilon$ terms to impose causal boundary conditions on the propagators, we find that $\Lambda'$ has an infinitesimal negative imaginary part.

Start evolution of $\Lambda'$ at large $\kappa$ from the asymptotic solution,

$$\kappa^2 \Lambda'(t) = \frac{1}{28} \left( 31 - 5 \sqrt{535} \tan \left[ \frac{1}{25} \left( 5 - \delta i - \sqrt{535} t \right) \right] \right),$$

with small a imaginary part $\delta$.

This allows our solution to bypass the singularities on the correct side.

Don’t take finite imaginary part to be too small relative to the numerical precision used in the integration! (See Schmidt and Moroz)
\( \bar{u}_2 \)

- The three-body parameter in spatially symmetric channel, \( \lambda' \), couples to the evolution of the dimer-dimer parameter in the corresponding four-body channel, \( \bar{u}_2 \).
- For large \( \kappa \gg 1 \), work with \( V_{DD}(\kappa) = \kappa^3 \bar{U}_2 \). See below.
- Apart from a short-lived transient determined by the driving term deriving from \( \Lambda' \). (\( V_{DD} \) is “irrelevant”.)
- See periodic singularities, reflecting the Efimov physics in the three-body channel.

The evolution of \( V_{DD} \) as a function of \( t \) in the scaling regime, starting with the initial condition \( V_{DD} = 0 \) at \( t = 100 \), for three different choices for the phase of \( \Lambda' \) on the asymptotic limit cycle (yellow dotted, red dashed, solid blue). The imaginary part of \( \Lambda' \) at the starting point is the same in the three cases shown.
Full results

- $V_{DD}$ goes to zero like $\kappa^3$ as $\kappa \to 0$.
- Suggests that the most appropriate technique is to use the equations for the rescaled quantities, $V_{DD}$ and $\kappa^2 \Lambda'$ for $t = \ln \kappa > t_0 \approx 0$, and the equations for $\bar{U}_2$ and $\Lambda'$ for $t < t_0$.
- Both couplings tend to finite values in the physical limit.
- The physical value of $\bar{U}_2$ is in general complex: Inelastic channel, as a result of the more deeply bound states in the three-body channel.
- Scattering lengths, and take $a_t = 4.32 \text{ fm}$ to reproduce the deuteron binding energy. Use the relations

\[
\frac{a_{DD}}{a_t} = 32 \pi \bar{U}_2(0), \\
\frac{a_{DN}}{a_t} = \frac{4}{3} \Lambda'(0),
\]
The relations between the scattering lengths in the SU(4) limit for the channels exhibiting the Efimov effect. All the points on these curves are obtained by choosing different points on the asymptotic limit cycle for $\Lambda'$ as initial values.
Mixed symmetry

- In the other pair of channels (mixed symmetry) all three- and four-body parameters are irrelevant.
- All the physical quantities can be related to $a_t$:

\[ a'_{DD}/a_t = 32\pi U_2(0) = 1.34, \]
\[ a'_{DN}/a_t = \frac{4}{3} \Lambda = \frac{5 \sqrt{215}}{7} - \frac{29}{3} = -0.43. \]

- Stable result: dimer-nucleon scattering length has the opposite sign to the nucleon-nucleon scattering length.
- There are thus two asymptotic solutions, both of which are negative.
- The sign of $\Lambda$ is preserved in the evolution out of the asymptotic regime, and hence, even for a different cut-off function, as long as we insist on a consistent $k$-scaling, the opposite sign remains.
Broken SU(4)

- Everything couples.
- Need to deal with $2 \times 2$ matrices of coupling constants in both the $(1/2, 1/2)$ three-nucleon channels and the $(0, 0)$ four-nucleon ones.
- Identify two scattering “eigenchannels” and, from the zero-energy $T$ matrix, determine a scattering length in each of these channels.
- Analyze the behaviour of these scattering lengths here, taking value of $\alpha = a_t/a_s = -1/4$ for the SU(4) breaking parameter. This is close to the realistic case, giving $a_s = -17.2\text{ fm}.$
Evolution of the eigenvalues of the $T$-matrix (parametrised as scattering lengths) in the two $B = 3$ $(1/2, 1/2)$ channels, and similar for the two $B = 4$ $(0, 0)$ channels, as a function of the parameter $\phi_0$ specifying the initial condition on the limit cycle.
**SU(4) symmetry?**

- The narrowness of the avoided crossings is an indication that SU(4)-breaking effects are relatively weak.
- Further evidence of the smallness of this breaking is provided by the ratios between the components of the eigenchannel solutions.
- As can be seen from below, the mixings are small ($\lesssim 20\%$), except in narrow windows around the crossing points.

**Graphical Representations:**

![Graph](image.png)

*Ratio between components of the couplings in the eigenchannels for (left): $\lambda^{1/2,1/2}$ and (right): $\tilde{u}_2$.**
Efimov dominated

In the scattering length in these channels vary rapidly with the three-body parameter. This means that at the avoided crossing, we switch from one branch of the solution to the other. As a result there is a very small discontinuity in the plots, close to the points where $a_{DN}$ and $\text{Re}[a_{DD}]$ vanish. In the other channels we have an approximately constant scattering lengths away from the avoided crossing. These have values of $a_{DN} = -0.26 \pm 0.03 \text{ fm}$ and $\text{Re}[a_{DD}] = 5.55 \pm 0.11 \text{ fm}$. 

The relations between the scattering lengths in the Efimov-dominated channels for $\alpha = -1/4$. 
Realistic scattering lengths

- We need one piece of three-body data to fix the starting point of the evolution on the Efimov cycle; We use experimental value of the spin-doublet $nd$ scattering length $2a^D_{nd} = 0.68$ fm to determine the initial value of $\lambda_t^{(1/2,1/2)}$. 
Spin Quartet channel

- The situation is different for the \( nd \) scattering length \( 4a_{nd} \) in the spin-quartet channel.
- All the three-body parameters are irrelevant and \( 4a_{nd} \) extracted at \( k = 0 \) does not depend on their initial values.
- Our FRG calculations give \( 4a_{nd} = -1.02 \) fm.
- This differs significantly from the experimental value of \( 4a_{nd} = 6.35 \) fm (W. Dilg et al).
- Other theoretical calculations seem to be able to reproduce this result, either by (Ruprecht) or by solving the Skornyakov–Ter-Martirosyan equation (Bedaque and Van Kolck).
- The negative value of this scattering length appears to be a stable result of our calculations.
- One possible reason for this is that the value of \( 4a_{nd} \) reported here is defined at \( E_D/2 \approx -1 \) MeV.
- Energy dependence is known to be important for three-body observables (Diehl) and so that the extrapolation to the physical threshold could have a significant effect, especially since \( E_d \) is a small scale, we could expect significant energy dependence.
- The work of Bedaque et al shows energy dependence, but when they include only the scattering length, this would probably not be sufficient to explain what happens here. In order to explore this in more detail, we will need to extend our REA to include energy/momentum-dependent couplings.
Four nucleon channel

- Extract the $dd$ scattering lengths in the spin-singlet and quintet channels. The low-energy $dd$ interaction has astrophysical applications: inside brown-dwarf stars a many-deuteron system may behave as a superfluid (Berezhiani et al).

- Expansion point corresponds to threshold. The singlet channel can couple to the $n+^3\text{He}$ (or $p+^3\text{H}$) channel which has a lower threshold. Scattering can be inelastic, scattering length is complex.

- The coupling of the quintet channel to the rearrangement ones is much smaller as non-zero orbital angular momentum is required. In our treatment, this channel is closed.

- Singlet $dd$ scattering length with $\text{Re}[^1a_{dd}] = 4.44 \text{ fm}$ and $\text{Im}[^1a_{dd}] = 0.17 \text{ fm}$. The real part is consistent with the value of $\text{Re}[^1a_{dd}] = 4.9 \text{ fm}$ obtained by solving the Faddeev–Yakubovsky equation (F. Ciesielski, J. Carbonell); imaginary part is larger.

- $\text{Re}[^5a_{dd}] = 2.55 \text{ fm}$. Agrees with the value $\text{Re}[^5a_{dd}] = 3.2 \text{ fm}$ obtained by Rupak,

- What is more puzzling is that Rupak finds a very different value for $^4a_{nd} = 4.78 \text{ fm}$, which feeds into the $(2,0)$ four-body one.

- Yet another suggestion that energy dependence of these couplings needs to be considered?
Quintet channel results agree qualitatively with the exact quantum mechanical analysis in (I.N. Filikhin and S. L. Yakovle), given our incomplete treatment of the 3+1-particle rearrangement channels. The authors of that work find that this scattering length is very sensitive to these channels, and excluding them leads to a substantial reduction of $\text{Re}[^5a_{dd}]$, from 7.5 fm to $-0.1$ fm.

At this level we do not describe the nucleon-trimer threshold that is needed to get the correct energy dependence in this channel. To do this we would need to extend our approach by adding an auxiliary trimer field, along the lines suggested by Schmidt and Moroz. This leads to a much larger set of coupled evolution equations. [Jaramillo]
We have applied the FRG method to three- and four-nucleon systems, for Wigner SU(4) symmetry, and with realistic symmetry breaking.

- We have calculated the nucleon-deuteron and deuteron-deuteron scattering lengths in various spin-isospin channels.

Evolution of one three-body coupling shows oscillatory, limit-cycle behaviour which is a manifestation of the Efimov effect in the corresponding channel.

- One piece of data to fix one three-body parameter; all other three- and four-nucleon observables are then predicted in terms of this and the two-body scattering lengths.
- The observables in mixed symmetry channels are independent of the initial scale; only depend on two-body scattering lengths

The Efimov effect appears in the spin-doublet nucleon-deuteron and the singlet deuteron-deuteron channels.

- We get a value for the singlet deuteron-deuteron scattering length that is very close to one obtained in the exact quantum mechanical calculations.
- Vvalue for the spin-quintet scattering length shows no significant dependence on the three-body parameter, but differs from the results of other calculations.
There is a variety of ways in which the present study could be improved.

- One important one is the introduction of a trimer field in order to describe better the nucleon-trimer channel, which is responsible for the inelasticity in deuteron-deuteron scattering. [Talk by Ben Jaramillo]
- A second is to extend the ansatz for the running action to include energy- or momentum-dependent couplings.


