Block Decimation Renormalization Group and Finite Range Scaling Analysis of Dissipative Double Well Quantum Mechanics.

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In classical mechanics, we can treat dissipation by modifying equation of motion to add a friction term, e.g., proportional to velocity. In the usual Quantum mechanical regulation, we need to define Hamiltonian first, but there is no simple Hamiltonian to realize the dissipative effects.
Caldeira-Leggett model

Dissipation is formulated from microscopic origin.

\[
S[q, \{x_\alpha\}] = \int dt \left\{ \frac{1}{2}M\dot{q}^2 - V_0(q) + \sum_\alpha \left[ \frac{1}{2}m_\alpha \dot{x}_\alpha^2 - \frac{1}{2}m_\alpha \omega_\alpha^2 x_\alpha^2 - qC_\alpha x_\alpha \right] \right\}
\]

path integrate out environmental degrees of freedom

non-local effective action in the imaginary time direction

\[
\Delta S_{NL} = \frac{\eta}{4\pi} \int ds \int d\tau \frac{(q(s) - q(\tau))^2}{|s - \tau|^{p}}
\]

Taking the Euclidean path integral formalism, the quantum mechanical system is equivalent to the 1-D statistical system.

Simplest approx.: Discretizing time and taking only two values \( \sigma = \pm v \) for \( q \), the system is equivalent to 1-D Effective Ising spin model with long range interactions.

The series of CL interactions \( K_n \) at a distance of \( n \) sites.

\[
H = \eta \sum_{ij} \frac{(\sigma_i - \sigma_j)^2}{(i - j)^p} = -\sum K_n^{[p]} \sigma_i \sigma_{i+n} \quad K_n^{[p]} \equiv \frac{\eta}{n^p}
\]
In particular, long range Ising spin model has a long history itself and it is known that phase transition exists for $1 < p \leq 2$.

- Dyson (1969), Griffiths (1967), Ruelle (1968)
- Frohlich and Spencer (1982)
- Kosterlitz (1976), Candy (1981)
- Aizenman, Chayes, Chayes, and Newman (1988)
- Imbie and Newman (1988)

Our scheme to approach infinitely long range interactions is as follows:

BDRG (Block Decimation Renormalization Group)

Calculation for the finite range interactions.

FRS (Finite Range Scaling method)

Evaluation of the criticality.

Quantitative determination of the critical dissipation $\eta_c$

The scaling exponent $\beta$ determines the phase transition point.
Recall DRG (Decimation Renormalization Group)  

Wilson 1975

1-D nearest neighbor Ising spin model

\[ H = - \sum K_1 \sigma_i \sigma_{i+1} \]

nearest neighbor interaction : \( K_1 \)

\[ h \] : external field

Boltzmann weight (decimation)

\[ \sum_{\sigma_2 = \pm 1} e^{K_1 \sigma_1 \sigma_2} e^{K_1 \sigma_2 \sigma_3} = e^{K_1 (\sigma_1 + \sigma_3)} e^{-K_1 (\sigma_1 + \sigma_3)} = 2 \cosh \{ K_1 (\sigma_1 + \sigma_3) \} \]

Initial Transfer matrix (represents the interactions between neighboring sites.)

\[ T^{(0)} = \begin{pmatrix} \exp(K_1 + h) & \exp(-K_1) \\ \exp(-K_1) & \exp(K_1 - h) \end{pmatrix} \]

\( h \): external field

k-th RG transformation (decimation)

\[ T^{(k)} \equiv T^{(k-1)} T^{(k-1)} \]
Physical quantities after $k$-th renormalization

- Partition function: $Z^{(k)} = \text{Tr} \ T^{(k)}$

- Free energy (per site): $F^{(k)} = \frac{1}{2^k} \log \text{Tr} \ T^{(k)}$

- Susceptibility:
  \[ \chi^{(k)} = \frac{\partial^2}{\partial h^2} F^{(k)} \bigg|_{h=0} \]

* Susceptibility is expected to diverge at the phase transition point with infinitely long range interactions. This will tell us the critical dissipation.
BDRG (Block Decimation Renormalization Group)

- Non-nearest interactions are not easily treated by the original DRG because it requires the interaction space of infinite dimension.
- We define BDRG, an extended DRG to fit long range (but finite) interactions.
- We set a maximal range of interaction to be $n$ and define a block of size $n$.
- Then, there are only nearest neighbor inter-block interactions. The system is regarded as a nearest neighbor multi-state model.

$$H = - \sum K_n^{[p]} \sigma_i \sigma_{i+n} \quad K_n^{[p]} = \frac{\eta}{n^p} = \frac{K_1}{n^p}$$

<example $n=3$>

The $k$-th RG transformation

$$T^{(k)} = T^{(k-1)} T^{(k-1)}$$
• $T$ (transfer) matrix represents the interactions between neighboring blocks.
• In case of long range Ising model, the number of states in a block is $2^n$ and the dimension of $T$ matrix is $2^n \times 2^n$.

$T$ matrix for example $n=2$

$$
\begin{pmatrix}
\uparrow\uparrow\cdot\uparrow\uparrow & \uparrow\uparrow\cdot\uparrow\downarrow & \uparrow\uparrow\cdot\downarrow\uparrow & \uparrow\uparrow\cdot\downarrow\downarrow \\
\uparrow\downarrow\cdot\uparrow\uparrow & \uparrow\downarrow\cdot\uparrow\downarrow & \uparrow\downarrow\cdot\downarrow\uparrow & \uparrow\downarrow\cdot\downarrow\downarrow \\
\downarrow\uparrow\cdot\uparrow\uparrow & \downarrow\uparrow\cdot\uparrow\downarrow & \downarrow\uparrow\cdot\downarrow\uparrow & \downarrow\uparrow\cdot\downarrow\downarrow \\
\downarrow\downarrow\cdot\uparrow\uparrow & \downarrow\downarrow\cdot\uparrow\downarrow & \downarrow\downarrow\cdot\downarrow\uparrow & \downarrow\downarrow\cdot\downarrow\downarrow
\end{pmatrix}
$$

$$
\begin{pmatrix}
e^A & e^F & e^E & e^B \\
e^E & e^C & e^D & e^F \\
e^F & e^D & e^C & e^E \\
e^B & e^E & e^F & e^A
\end{pmatrix}
$$

*A~F are appropriate linear combinations of spin coupling constants. In this case, there are 6 independent interactions between sites.

• BDRG is able to numerically calculate finite-range system exactly.
• There are systematic approximations to exact BDRG with smaller Dimensional RG flow space. (No time to discuss its details here.)
Finite Range Scaling (FRS) method

- This is a new method to analyze infinite range system using only finite range information.
- First, the finite range susceptibility $\chi(n)$ is calculated exactly by BDRG.
- Next, assuming that the variation of the susceptibility with respect to range $n$ satisfies the following scaling relation, we define the scaling exponent $\beta$ (FRS exponent),

$$\Delta(n, p, \eta) \equiv \frac{1}{2\eta} \left( \log \chi(n) - \log \chi(n - 1) \right) \equiv \left( \frac{1}{n} \right)^{\beta(n, p, \eta)}$$
• The infinite $n$ behavior of $\chi(n)$ is controlled by the zeta function $\zeta(\beta)$.

$$\lim_{n \to \infty} \log \chi = 2\eta \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\beta(\infty, p, \eta)} + \text{finite} = 2\eta \zeta[\beta(\infty, p, \eta)] + \text{finite}$$

• $\zeta(\beta)$ has a pole singularity at $\beta=1$:

$$\zeta(\beta) \approx \frac{1}{\beta - 1}$$

• $\beta$ moves from $p$ to $p-1$

$$\beta(n, p, \eta) = \begin{cases} p & \text{small } \eta \text{ (Weak)} \\ p - 1 & \text{large } \eta \text{ (Strong)} \end{cases}$$

• Finally, the critical $\eta$ is determined by the condition:

$$\beta(p, \eta_c) = 1$$
Behavior of FRS exponent $\beta$

### Graph Details:
- **Behavior of $\beta$**
- **$\beta = 1$**
- **$p = 1.8$**
- **$\eta_c$**

### Axes:
- $\beta$ on the y-axis
- $\eta$ on the x-axis

### Key Points:
- The graph shows the behavior of $\beta$ as $\eta$ varies.
- The value of $\beta$ decreases as $\eta$ increases.
- The critical point $\eta_c$ is indicated on the x-axis.
Results of critical dissipation (long range Ising)

Our FRS results
Aoki, Kobayashi and Tomita 08

Luijten, Blote 97 and Luijten, Messingfeld 01

Dyson 69, Griffiths 67

Aoki, Kobayashi and Tomita (2008)
Our methods have been successful in the long range Ising model. Now, is it effective for dissipative double well quantum mechanics?

**BDRG**
(Block Decimation Renormalization Group)

**FRS**
(Finite Range Scaling method)

Quantitative determination of the critical dissipation \( \eta_c \)

The scaling exponent \( \beta \) determines the phase transition point.

Calculation for the finite range interactions.

Evaluation of the criticality.

CL long range interactions between sites in discretized quantum mechanics

\[
\frac{\eta}{4\pi} \epsilon^2 \sum_{i,j} \frac{(q_i - q_j)^2}{|i - j|^p \epsilon^p} = \frac{\eta}{2\pi} \epsilon^{2-p} \sum_n \frac{(q_i - q_{i+n})^2}{n^p}
\]
We restrict interaction range \( n \), then **1-block involves \( n \) sites.**

Note that a state is infinite dimensional in quantum mechanics.

\[
\begin{align*}
&T = e^{-W(x_1 \cdots x_n, y_1 \cdots y_n)} = \langle x_1 x_2 \cdots x_n | \hat{U} | y_1 y_2 \cdots y_n \rangle \\
&\text{completeness: } \int dy_1 \cdots dy_n | y_1 \cdots y_n \rangle \langle y_1 \cdots y_n | = 1 \\
&= \sum_{a_1 \cdots a_n} | a_1 \cdots a_n \rangle \langle a_1 \cdots a_n | \\
&\text{*We need to choose a proper set of states.}
\end{align*}
\]

\[
\begin{align*}
&T = e^{-\hat{W}(a_1 \cdots a_n, b_1 \cdots b_n)} = \langle a_a \cdots a_n | \hat{U} | b_1 \cdots b_n \rangle \\
&= \int dx_1 \cdots dx_n dy_1 \cdots dy_n \langle a_1 \cdots a_n | x_1 \cdots x_n \rangle \langle x_1 \cdots x_n | \hat{U} | y_1 \cdots y_n \rangle \langle y_1 \cdots y_n | b_1 \cdots b_n \rangle \\
&= \int dx_1 \cdots dx_n dy_1 \cdots dy_n \psi_{a_1}(x_1) \cdots \psi_{a_n}(x_n) e^{-W(x_1 \cdots x_n, y_1 \cdots y_n)} \psi_{b_1}(y_1) \cdots \psi_{b_n}(y_n)
\end{align*}
\]

States \( \{ | a_n \rangle \} \) need to be restricted.
Ground state approximation

We take only two states, the linear combination of ground state and 1st excited state of the double well without dissipation, $\eta = 0$.

Our initial aim is to evaluate the feasibility of our BDRG & FRS in the effective Ising model given by the ground state approximation.

$$\psi_{\uparrow, \downarrow} = (|0\rangle \pm |1\rangle)/\sqrt{2}$$

$\lambda = 0.04$

2-state approximation

Our initial aim is to evaluate the feasibility of our BDRG & FRS in the effective Ising model given by the ground state approximation.
For example, $n=2$

\[ \{ \psi_{a_n}(x) \} = \{ \psi_\uparrow(x), \psi_\downarrow(x) \} \quad \text{ground state approximation} \]

\[ T = e^{-W(a_1 a_2, b_1 b_2)} = \int dx_1 dx_2 dy_1 dy_2 \ \psi_1^*(x_1) \psi_2^*(x_2) \psi_1(y_1) \psi_2(y_2) \]

\[ \times \exp \left[ -\frac{m(x_1 - x_2)^2}{4\epsilon} - \frac{m(x_2 - y_1)^2}{2\epsilon} - \frac{m(y_1 - y_2)^2}{4\epsilon} \right] : \text{kinetic term} \]

\[ -\frac{\epsilon}{2} (V(x_1) + V(x_2) + V(y_1) + V(y_2)) : \text{potential term} \]

\[ -\frac{\eta}{2\pi \epsilon^{2-p}} \left[ \frac{1}{2} (x_1 - x_2)^2 + (x_2 - y_1)^2 + \frac{1}{2} (y_1 - y_2)^2 \right] : \text{dissipation term} \]

\[ T = e^{-W(x_1 x_2, y_1 y_2)} \]

- $2^4$ integrations of 4-dimension
- In the ground state approximation, $2^{2n}$ integrations of $2n$-dimensions are necessary to get the initial $T$ matrix of BDRG. To make highly multi-dimensional integrations, we adopt the Monte Carlo method (heat bath method).
Behavior of $\beta$

rough: $p = 1.99$, $\epsilon = 0.9$, $\lambda = 0.04$, $N_{MC} = 128 \times 10^4$, 16 seeds

$n=5$
$n=6$
$n=7$

Statistical error

\[ \sigma_\beta = \frac{s_\beta}{\sqrt{\text{seed number}}} \]

$s_\beta$: the standard deviation of $\beta$ which is evaluated from the average $\beta$ of each seed.
The diagram shows the region of β7-detailed criticality with the following parameters:

- $p = 1.99$, $\epsilon = 0.9$, $\lambda = 0.04$, $N_{MC} = 1024 \times 10^4$, 64 seeds

The shaded area represents the ±σ region around $\eta_c$. The criticality is indicated by the green line at $\eta = 1$. The x-axis represents $\eta$ from 0.4 to 0.54.
Results of critical dissipation (QM)\[ V(q) = -\frac{1}{2}q^2 + \lambda q^4 \]

\[ \eta_c \]

- MC
- Instanton
- NPRG-WH
- FRS

\[ \lambda \]
Summary

- We study the Classical-Quantum phase transition in the dissipative double well quantum mechanics where the Caldeira-Leggett effective long range interactions are introduced for the dissipative effects.
- Discretizing the time and taking the 2-state approximation using the ground state, we convert the system into an effective long range Ising model. We apply our BDRG & FRS method, which has been proven successful in the simple long range Ising model, to this effective Ising model.
- Highly multi-dimensional integrations are necessary to calculate effective Ising interactions and we adopt the Monte Carlo integration.
- Then applying our FRS method, we get the critical dissipation by the condition, the FRS exponent $\beta (\eta_c) = 1$.
- Our results consistently give interpolating values between Monte Carlo and the Instantons.
- Now we pursue full-BDRG without ad-hoc 2-state approximation, where the decimation renormalization itself is implemented directly by Monte Carlo method.